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THE PERIMETER OF AN ELLIPSE

TIRUPATHI R. CHANDRUPATLA * AND
THOMAS J. OSLER,** *Rowan University*

Abstract

There is no simple way to calculate the perimeter of an ellipse. In this expository paper we review and compare four methods of evaluating this perimeter. These methods are known by the names Maclaurin, Gauss–Kummer, Cayley, and Euler. Maclaurin's method is derived in detail, while the other three are simply described. A short introduction to hypergeometric functions is included, since these functions are useful for finding the elliptic perimeter. We compare the usefulness of these four calculations, An appendix gives MICROSOFT EXCEL® programs for computing the perimeter.

Keywords: Perimeter of ellipse; hypergeometric functions

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1. Introduction

The area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by the formula $A = \pi ab$. We recognize this as a simple generalization of the formula for the area of a circle of radius a given by $A = \pi a^2$. What can we say about the perimeter of this ellipse? The circle of radius a has circumference $C = 2\pi a$, so is the perimeter of the ellipse something simple like $P = 2\pi(a+b)/2$, where we have used the average of a and b in place of the radius of the circle? The answer is no.

Unfortunately, there is no simple way to express the perimeter of the ellipse in terms of elementary functions of a and b. To express this perimeter we need to expand our tool kit of functions beyond the trigonometric, exponential, and logarithmic functions studied in the calculus. We shall require 'special functions' such as the gamma function, the elliptic functions, and the hypergeometric functions, just to name a few. In this paper we will take a quick look at 'complete elliptic integrals' and 'hypergeometric functions' in our quest to calculate the perimeter of the ellipse.

In Sections 2 and 3 we will discuss in detail one method of calculating the perimeter known as the Maclaurin expansion (see [5])

$$P = 2\pi a\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right],$$

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* Postal address: Mechanical Engineering Department, Rowan University, Glassboro, NJ 08028, USA.
Email address: chandrupatla@rowan.edu
** Postal address: Department of Mathematics, Rowan University, Glassboro, NJ 08028, USA.
Email address: osler@rowan.edu

where $k^2 = 1 - b^2/a^2$. This result will also be expressed in terms of elliptic integrals and hypergeometric functions in Section 4. In Sections 5 and 6 we take a quick look at some properties of hypergeometric functions, and in Section 7 we introduce three additional formulas for finding the perimeter of an ellipse without giving their derivation. In Section 8 we use the results of computer calculations to compare the four formulas discussed in the paper. Appendix A gives MICROSOFT EXCEL programs for computing the perimeter.

2. The perimeter and elliptic integrals

The differential arc length for a curve given by parametric equations $x = x(\theta)$ and $y = y(\theta)$ is

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

The ellipse given by the parametric equations $x = a\cos\theta$ and $y = b\sin\theta$ has differential arc length

$$\mathrm{d}s = \sqrt{(-a\sin\theta)^2 + (b\cos\theta)^2}\,\mathrm{d}\theta.$$

Therefore, the perimeter of the ellipse is given by the integral

$$P = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,\mathrm{d}\theta,$$

in which we have quadrupled the arc length found in the first quadrant. Replacing $\sin^2\theta$ by $1 - \cos^2\theta$ we get

$$\begin{aligned} P &= 4\int_0^{\pi/2} \sqrt{a^2(1-\cos^2\theta) + b^2\cos^2\theta}\,\mathrm{d}\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\cos^2\theta}\,\mathrm{d}\theta. \end{aligned} \tag{1}$$

If we let

$$k^2 = 1 - \frac{b^2}{a^2}, \tag{2}$$

we can write (1) as

$$P = 4a\int_0^{\pi/2} \sqrt{1 - k^2\cos^2\theta}\,\mathrm{d}\theta. \tag{3}$$

Since $\cos\theta = \sin(\pi/2 - \theta)$, for any function $f(x)$ we can write

$$\int_0^{\pi/2} f(\cos\theta)\,\mathrm{d}\theta = \int_0^{\pi/2} f\left(\sin\left(\frac{\pi}{2} - \theta\right)\right)\mathrm{d}\theta.$$

With the substitution $t = \pi/2 - \theta$ we convert this to

$$\int_0^{\pi/2} f(\cos\theta)\,\mathrm{d}\theta = \int_0^{\pi/2} f(\sin t)\,\mathrm{d}t.$$

Therefore, we can replace the cosine in (3) by a sine to obtain

$$P = 4a\int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\,\mathrm{d}\theta.$$

The reason for this conversion is to express the perimeter in terms of the named function

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta, \tag{4}$$

which is called a *complete elliptic integral of the second kind.* Thus we can now write

$$P = 4aE(k).$$

The *complete elliptic integral of the first kind* defined by the integral

$$F(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

occurs in the analysis of the motion of the pendulum with large amplitude [7, pp. 331–335].

3. Maclaurin series expansion for the perimeter

Unfortunately we cannot express the elliptic integral (4) in terms of elementary functions. To assist in the integration of (4), we begin by expanding the square root in the integrand using the binomial theorem. We have

$$\begin{aligned} \sqrt{1-x} &= (1-x)^{1/2} \\ &= \sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} (-1)^j x^j, \end{aligned} \tag{5}$$

in which the binomial coefficient

$$\binom{p}{j} = \frac{p}{1}\,\frac{p-1}{2}\,\frac{p-2}{3} \cdots \frac{p-j+1}{j}$$

is valid for $j = 1, 2, 3, \ldots$, while $\binom{p}{0} = 1$. Thus, for $j = 1, 2, 3, \ldots$, we have

$$\begin{aligned} (-1)^j \binom{\frac{1}{2}}{j} &= (-1)^j \frac{\frac{1}{2}}{1}\,\frac{\frac{1}{2}-1}{2}\,\frac{\frac{1}{2}-2}{3}\,\frac{\frac{1}{2}-3}{4} \cdots \frac{\frac{1}{2}-j+1}{j} \\ &= \frac{-\frac{1}{2}}{1}\,\frac{1-\frac{1}{2}}{2}\,\frac{2-\frac{1}{2}}{3}\,\frac{3-\frac{1}{2}}{4} \cdots \frac{j-1-\frac{1}{2}}{j}. \end{aligned}$$

We can rewrite this as

$$(-1)^j \binom{\frac{1}{2}}{j} = -\frac{1}{2}\,\frac{1}{4}\,\frac{3}{6}\,\frac{5}{8} \cdots \frac{2j-3}{2j}.$$

Returning to (5) we can now write

$$\sqrt{1-x} = 1 - \sum_{j=1}^{\infty} \frac{1}{2}\,\frac{1}{4}\,\frac{3}{6}\,\frac{5}{8} \cdots \frac{2j-3}{2j} x^j.$$

The complete elliptic integral (4) can be written as

$$E(k) = \int_0^{\pi/2} \left(1 - \sum_{j=1}^{\infty} \frac{1}{2}\,\frac{1}{4}\,\frac{3}{6}\,\frac{5}{8} \cdots \frac{2j-3}{2j} k^{2j} \sin^{2j} \theta \right) d\theta.$$

Integrating term by term we have

$$E(k) = \frac{\pi}{2} - \sum_{j=1}^{\infty} \frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2j-3}{2j}k^{2j}\int_0^{\pi/2}\sin^{2j}\theta\,\mathrm{d}\theta. \tag{6}$$

We will make use of the known definite integral (available from software such as MATHEMATICA® or standard mathematical tables)

$$\int_0^{\pi/2}\sin^{2j}\theta\,\mathrm{d}\theta = \frac{1\cdot 3\cdot 5\cdots(2j-1)}{2\cdot 4\cdot 6\cdots 2j}\,\frac{\pi}{2}$$

to rewrite (6) as

$$\begin{aligned} E(k) &= \frac{\pi}{2}\left[1 - \sum_{j=1}^{\infty}\left(\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2j-3}{2j}\right)\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2j-1}{2j}\right)k^{2j}\right] \qquad (7)\\ &= \frac{\pi}{2}\left[1 - \sum_{j=1}^{\infty}\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2j-1}{2j}\,\frac{1}{2j-1}\right)\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2j-1}{2j}\right)k^{2j}\right]. \end{aligned}$$

This last relation is the Maclaurin series expansion for the complete elliptic integral

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2\frac{k^2}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^6}{5} - \cdots\right].$$

4. Hypergeometric functions and the perimeter

We can also express the perimeter of the ellipse in terms of hypergeometric functions which are defined by

$$F(\alpha,\beta;\gamma;x) = \sum_{j=0}^{\infty}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j\,j!}x^j, \qquad |x| < 1,$$

where we have used the Pochhammer symbol

$$(\alpha)_j = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+j-1).$$

Rearranging (7)

$$E(k) = \frac{\pi}{2}\left[1 + \sum_{j=0}^{\infty}\left(-\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2j-3}{2j}\right)\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2j-1}{2j}\right)k^{2j}\right], \tag{8}$$

we rewrite the first factor in the sum in (8) as

$$\begin{aligned} -\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2j-3}{2j} &= \frac{-\frac{1}{2}}{1}\,\frac{1-\frac{1}{2}}{2}\,\frac{2-\frac{1}{2}}{3}\,\frac{3-\frac{1}{2}}{4}\cdots\frac{j-1-\frac{1}{2}}{j}\\ &= \frac{(-\frac{1}{2})_j}{j!}. \qquad (9) \end{aligned}$$

The second factor in the sum in (8) can also be expressed in terms of Pochhammer symbols as

$$\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2j-1}{2j}=\frac{\frac{1}{2}}{1}\frac{\frac{1}{2}+1}{2}\frac{\frac{1}{2}+2}{3}\cdots\frac{\frac{1}{2}+j-1}{j}$$
$$=\frac{(\frac{1}{2})_j}{j!}. \tag{10}$$

Replacing the factors in (8) by (9) and (10) we can now write

$$E(k)=\frac{\pi}{2}\left[1+\sum_{j=0}^{\infty}\frac{(-\frac{1}{2})_j}{j!}\frac{(\frac{1}{2})_j}{j!}k^{2j}\right],$$

which can be expressed in terms of the hypergeometric function as

$$E(k)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{11}$$

so that the perimeter of the ellipse is

$$P=4aE(k)=2a\pi F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right).$$

5. Hypergeometric function identities

In this section we summarize some facts concerning the important hypergeometric functions without giving their derivations.

The hypergeometric function $y=F(\alpha,\beta;\gamma;z)$ is the solution of the differential equation

$$z(1-z)y''+[\gamma-(\alpha+\beta-1)z]y'-\alpha\beta y=0.$$

The following elementary functions can be expressed in terms of hypergeometric functions:

$$(1+x)^p=F(-p,1;1;-x),$$
$$(1-x^2)^{1/2}=F\left(-\frac{1}{2},1;1;x^2\right),$$
$$\ln(1+x)=xF(1,1;2;-x),$$
$$\ln\left(\frac{1+x}{1-x}\right)=2xF\left(\frac{1}{2},1;\frac{3}{2};x^2\right),$$
$$\arcsin x=xF\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right),$$
$$\arctan x=xF\left(\frac{1}{2},1;\frac{3}{2};-x^2\right).$$

We have shown that the complete elliptic integral of the second kind can be expressed in terms of the hypergeometric function (11), and we can also express the complete elliptic integral of the first kind as

$$F(k)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right).$$

The following are known transformations for the hypergeometric function:

$$F\left(\alpha, \beta; 2\beta; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha} F\left(\alpha, \alpha - \beta + \frac{1}{2}; \beta + \frac{1}{2}; z^2\right), \tag{12}$$

$$F(\alpha, \beta; 2\beta; 2z) = (1-z)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \beta + \frac{1}{2}; \frac{z^2}{(1-z)^2}\right). \tag{13}$$

A list of notable mathematicians who have contributed to the theory of the hypergeometric function include Colin Maclaurin (1698–1746), Leonhard Euler (1707–1783), John Landen (1719–1790), James Ivory (1765–1842), Carl Friedrich Gauss (1777–1855), Ernst Eduard Kummer (1810–1893), and Arthur Cayley (1821–1895).

6. Generalized hypergeometric functions

In this section we briefly introduce the generalized hypergeometric function defined by the series

$$_mF_n\left(\begin{matrix} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_n \end{matrix} \,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_m)_k}{(b_1)_k (b_2)_k \cdots (b_n)_k} \frac{x^k}{k!}.$$

If $m \le n$ the series converges for all x, if $m = n+1$ the series converges for $|x| < 1$, and if $m > n+1$ then the series diverges. Another notation in common use is

$$_mF_n\left(\begin{matrix} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_n \end{matrix} \,\middle|\, x\right) = {}_mF_n(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; x).$$

The ordinary hypergeometric function considered in previous sections can thus be written as

$$F(a, b; c; x) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, x\right) = {}_2F_1(a, b; c; x).$$

All of the formulas listed in Section 5 without derivation can be found in [1].

7. Three additional formulas for the perimeter

In this section we list three additional formulas for the perimeter of the ellipse without giving their derivation. Two of the three can be expressed conveniently in terms of the hypergeometric function.

Our first expression is the *Gauss–Kummer relation*

$$P = \pi(a+b)\left[1 + \left(\frac{1}{2}\right)^2 h^2 + \left(\frac{1}{2\cdot 4}\right)^2 h^4 + \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}\right)^2 h^6 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\right)^2 h^8 + \cdots\right],$$

where

$$h = \frac{a-b}{a+b}.$$

We note that the Gauss–Kummer relation is derived from the hypergeometric series [2], [4]

$$P = \pi(a+b) F\left(-\frac{1}{2}, -\frac{1}{2}; 1; h^2\right). \tag{14}$$

The *Gauss–Kummer relation* (14) is obtained by setting $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, and $z = h = (a-b)/(a+b)$ in (12).

Our second formula was derived by Cayley [3, pp. 46–55] for the case when a/b is large by substituting $k = b/a$ in the governing differential equation. *Cayley's series* is

$$P = 4a\left[1 + \frac{1}{2}\left(\ln\frac{4}{k} - \frac{1}{1\cdot 2}\right)k^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)k^4 \right.$$
$$\left. + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)k^6 + \cdots\right]. \tag{15}$$

Our third formula was given by *Euler* and is represented by the hypergeometric relation

$$P = \pi\sqrt{2(a^2+b^2)}\,F\left(\frac{1}{4}, -\frac{1}{4}; 1; \left(\frac{a^2-b^2}{a^2+b^2}\right)^2\right). \tag{16}$$

Euler's formula (16) is obtained by setting $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, and $2z = k^2 = 1 - b^2/a^2$ in (13).

8. Comparing the four formulas for the perimeter

We now use a computer to compare the following four formulas for the perimeter of the ellipse:

$$\text{Maclaurin} \qquad P = 2a\pi F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \qquad k^2 = 1 - \frac{b^2}{a^2},$$

TABLE 1.

a	b	Perimeter	Best formula	Gauss–Kummer	Cayley	Euler	Maclaurin
1	0.999 9	6.282 871 152	G–K	2	76 027	2	4
1	0.99	6.251 808 848	G–K	3	1 192	4	9
1	0.9	5.973 160 433	G–K	6	136	7	18
1	0.8	5.672 333 578	G–K	7	68	10	29
1	0.7	5.382 368 981	G–K	9	44	14	42
1	0.6	5.105 399 773	G–K	10	31	20	62
1	0.5	4.844 224 11	G–K	13	24	28	93
1	0.4	4.602 622 519	G–K	16	18	42	147
1	0.3	4.385 910 07	Cayley	21	14	73	260
1	0.2	4.202 008 908	Cayley	30	11	153	562
1	0.1	4.063 974 18	Cayley	56	8	549	2 027
1	0.05	4.019 425 619	Cayley	102	6	1 941	7 131
1	0.04	4.013 143 313	Cayley	124	6	2 907	10 646
1	0.03	4.007 909 45	Cayley	159	6	4 880	17 791
1	0.02	4.003 839 16	Cayley	225	5	10 074	36 454
1	0.01	4.001 098 33	Cayley	405	4	34 183	121 721
1	0.005	4.000 309 233	Cayley	723	4	112 844	393 112
1	0.004	4.000 205 049	Cayley	870	4	164 534	568 193
1	0.003	4.000 120 518	Cayley	1 100	4	265 848	906 354
1	0.002	4.000 568 07	Cayley	1 527	3	515 404	1 719 206
1	0.001	4.000 155 88	Cayley	2 643	3	1 520 644	4 816 476
1	0.000 1	4.000 002 02	Cayley	13 943	3	20 939 398	43 144 616
1	0.000 001	4.000 000 00	Cayley	80 386	2	–	–

Gauss–Kummer $P = \pi(a+b)F\left(-\frac{1}{2}, \frac{1}{2}; 1; h^2\right), \qquad h = \frac{a-b}{a+b},$

Cayley
$$P = 4a\left[1 + \frac{1}{2}\left(\ln\frac{4}{k} - \frac{1}{1\cdot 2}\right)k^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)k^4 + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6} \times\left(\ln\frac{4}{k} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)k^6 + \cdots\right], \qquad k = \frac{b}{a},$$

Euler $P = \pi\sqrt{2(a^2+b^2)}\,F\left(\frac{1}{4}, -\frac{1}{4}; 1; \left(\frac{a^2-b^2}{a^2+b^2}\right)^2\right).$

In Table 1 we list our results for a standard ellipse with semi-major axis $a = 1$ and semi-minor axis b ranging from 1 to 0. The perimeter is listed in the third column. In the last four columns we give the number of terms needed in the series to compute the perimeter numerically to eight decimal places.

Appendix A. MICROSOFT EXCEL implementation

A user-defined *function* in MICROSOFT EXCEL is easy to develop. To open the spreadsheet, press Alt-F11. You will see Microsoft Visual Basic Book 1. Click 'Insert' and select 'Module'. You get the Module 1 screen. Type in the functions in visual basic as given below. The functions will be available in the spreadsheet. The three functions developed are `hgs(a,b,c,z)`, `gauskum(a,b)`, `cayley(a,b)`, and `gkc(a,b)`. The first one, `hgs(a,b)`, calculates the hypergeometric series $F(a, b; c; z)$, the other three evaluate the ellipse perimeter (`gauskum(a,b)` using the Gauss–Kummer series (14), `cayley(a,b)` using the Cayley formula (15), and `gkc(a,b)` using the Gauss–Kummer or Cayley series so that the perimeter is evaluated using the least number of terms in the series).

```
Function hgs(a, b, c, z)
t = a * b / c * z
s = 1 + t
n = 1
Do
   a = a + 1
   b = b + 1
   c = c + 1
   n = n + 1
   t = t * a / c * b / n * z
   s1 = s
   s = s + t
Loop While s1 <> s
hgs = s
End Function

Function gkc(a, b)
   a = Abs(a): b = Abs(b)
   If a < b Then
      c = a: b = a: c = b
```

```
   End If
   If b = 0 Then
      gkc = 4 * a
      Exit Function
   End If
   If b / a > 0.25 Then
      gkc = gauskum(a, b)
   Else
      gkc = cayley(a, b)
   End If
End Function

Function gauskum(a, b)
   h = (a - b) / (a + b)
   t = 0.5 * h
   p = 1 + t * t
   u = 1
   Do
      t = t * u / (u + 3) * h
      p1 = p
      p = p + t * t
      u = u + 2
   Loop While p1 <> p
   gauskum = Application.Pi() * (a + b) * p
End Function

Function cayley(a, b)
   x = b / a
   t2 = Log(4 / x) - 0.5
   x = x * x
   t1 = 0.5 * x
   p = 1 + t1 * t2
   a1 = 1: a2 = 2
   s1 = a1 / a2
   s2 = 1 / a1 / a2
   Do
      t1 = t1 * s1
      t2 = t2 - s2
      a1 = a1 + 2
      a2 = a2 + 2
      s1 = a1 / a2
      s2 = 1 / a1 / a2
      t1 = t1 * x * s1
      t2 = t2 - s2
      p1 = p
      p = p + t1 * t2
    Loop While p1 <> p
    cayley = 4 * a * p
End Function
```

The calculation terminates when the addition of a term does not change the value for given machine precision. The precision is about 1e–12.

Michon [6] discussed the Gauss–Kummer and Cayley series; the article provides the QBASIC function `gk(h)` using a combination of Gauss–Kummer and Cayley formulas.

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